Equivalence of the generalised grid and projection methods for the construction of quasiperiodic tilings

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Abstract. The two main techniques for the generation of quasiperiodic tilings, de Bruijn's grid method and the projection formalism, are generalised. A very broad class of quasiperiodic tilings is obtained in this way. The two generalised methods are shown to be equivalent. The standard calculation of Fourier spectra is extended to the whole general class of tilings. Various applications are discussed.

1. Introduction

The recent discovery of a new phase of Al–Mn alloy (Shechtman *et al* 1984) exhibiting icosahedrally symmetric diffraction patterns has led physicists' interest to quasiperiodic tilings. Prototypes of quasiperiodic tilings are the famous Penrose patterns (Gardner 1977, Penrose 1979). For these, a detailed algebraic theory had been developed by de Bruijn (1981). He showed that these tilings can be constructed from the so-called 'pentagrid'. This construction has a nice geometric interpretation: a Penrose pattern can be viewed as the projection of a surface in a 5D cubic lattice onto a suitable plane.

de Bruijn's ideas were soon generalised by Kramer and Neri (1984). In their prophetic work, published well before Shechtman's discovery, they constructed 3D icosahedrally symmetric tilings by projecting a 3D lattice hypersurface in a 6D cubic lattice onto a suitable 3-space. These tilings are composed of the two rhombohedra discussed earlier by Mackay (1981).

After a first attempt by Levine and Steinhardt (1984) along slightly different lines, several authors (Elser 1985, Duneau and Katz 1985, Kalugin *et al* 1985) interpreted the results of Shechtman *et al* in the spirit of Kramer and Neri. All these authors made use of the fact that the projection formalism offers an ingenious way of calculating the Fourier spectra analytically. Also other structural properties are studied most conveniently in the projection framework. For instance, Elser (1985) was able to calculate the frequency of occurrence of any given finite subpattern. On the other hand, the grid technique is best suited for the construction of tilings on a computer, yielding an algorithm that is much more transparent than a direct application of the projection method.

In this paper we introduce generalisations of both the grid and the projection technique. In this way we are able to construct a broad class of quasiperiodic tilings, e.g. such with any desired point symmetry. For the projection method, similar proposals have been made already by Duneau and Katz (1985). The main point is then to prove the equivalence of the two generalised methods. Our proof is a constructive one. It thus allows us to switch from one scheme to another depending on which one is more

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convenient for the purpose. In this way, various techniques originally developed for one method also become available for the other. We believe that this will provide new and powerful tools for the study of quasiperiodic tilings.

This paper is organised as follows. After briefly reviewing the 'classical' grid and projection methods, we introduce our generalisations. Then the equivalence of the two generalised methods is proven. Using this proof, in § 5 the projection technique of calculating Fourier spectra is extended to the whole general class of quasiperiodic patterns constructed earlier. We close with two examples. First we discuss 2D tilings with *n*-fold point symmetry. Then a new principle is presented for the construction of quasiperiodic tilings containing large periodic regions.

2. de Bruijn's grid method

Let $\{g_i\}_{i=1,...,5}$ be a star of unit vectors pointing to the vertices of a regular pentagon. We call them *grid vectors*. The *pentagrid* (or simply *grid*) G_5 is defined as the union of five arrays of equidistant parallel lines orthogon 1 to the vectors g_i:

$$G_5 = \{x \in E^2 | x \cdot g_l - \gamma_l = k_l; \ l = 1, \ldots, 5, \ k_l \in \mathbb{Z}\}. \tag{1}$$

The *grid parameters* $\gamma_l \in \mathbb{R}$ are translations of the arrays relative to the origin[†]. We assume the γ_l to be such that nowhere do more than two grid lines intersect. Such grids are called *regular*.

To every point $y \in E^2$ we assign a vector $K(y) \in \mathbb{Z}^5$ through

$$K_i(y) = \lceil y \cdot g_i - \gamma_i \rceil \qquad i = 1, \ldots, 5 \tag{2}$$

where $\lceil x \rceil = \min\{n \in \mathbb{Z} | n \geq x\}$. Let x_0 be an intersection point of two grid lines and $U(x_0)$ a small neighbourhood of x_0 containing no other intersection points. From (1) and (2), it follows that on $U(x_0)$ the vector function $K(y)$ takes on four different values $K(x_0, j)$, $j = 1, \ldots, 4$. One readily notes that the four vectors

$$P(x_0, j) = \sum_{i=1}^{5} K_i(x_0, j) \cdot g_i \qquad j = 1, \ldots, 4 \tag{3}$$

point to the vertices of one of the two Penrose rhombuses. The set of rhombuses defined by all intersection points of the grid form a perfect tiling (de Bruijn 1981), i.e. one with no holes or overlaps.

3. The projection method

In this section we review the projection method as discussed by Duneau and Katz (1985). For simplicity, we first restrict ourselves to cubic lattices. More general ones will be included later in this paper.

Let $\mathcal{L}$ be the union of N arrays of hyperplanes in E^N:

$$\mathcal{L} = \{x \in E^N | x \cdot h_l - \gamma_l = k_l; \ l = 1, \ldots, N, \ k_l \in \mathbb{Z}\} \tag{4}$$

where $\{h_l\}_{l=1,...,N}$ is an orthonormal basis of E^N. It will turn out that the γ_i occurring in (4) can in fact be identified with the grid parameters introduced in the last section.

[†] Note that our γ differ in sign from those of de Bruijn (1981).

The points where N hyperplanes intersect are the vertices of a simple cubic lattice L given by

$$L = \{x \in E^N | x = \sum_{l=1}^{N} (\gamma_l + k_l) \cdot h_l, k_l \in \mathbb{Z}\}. \tag{5}$$

Clearly, the γ_l are the coordinates of the origin of L. Let E^D be a D-dimensional subspace of E^N. We define an open *strip S* around E^D by

$$S = \{x = x_\parallel + \sum_{i=1}^{N} \alpha_i h_{i,\perp} | x_\parallel \in E^D, \alpha_i \in]-1, 0] \forall i\} \tag{6}$$

where $h_{i,\perp}$ is the component of h_i orthogonal to E^D. In other words, S is generated by moving a unit cube along E^D in such a way that its corner with the *highest* coordinates is always contained in E^D.

From the construction of S it follows that for generic γ there is a unique D-dimensional lattice hypersurface Σ contained in S. The vertices of Σ are the corners with the *lowest* coordinates of those unit hypercubes of the lattice which have non-zero intersection with E^D. The tiling is then the orthogonal projection of Σ onto E^D. It is composed of the projections of the $\binom{N}{D}$ different unit D cells of the lattice.

The connection to the grid method consists in the following observations. The grid G_N has to be identified with the intersection $\mathscr{L} \cap E^D$†. $\mathscr{L}$ divides E^D into open regions, and with everyone of these we can associate a cube of the lattice intersecting with E^D. The projections of the corners with the lowest coordinates of these cubes can then be calculated. This is precisely what is done in the grid method.

de Bruijn's Penrose pattern constructed in the preceding section can be obtained by choosing $N = 5$, $D = 2$, and by properly embedding the subspace E^2 in E^5. For details see § 8 or the original paper of de Bruijn (1981).

4. The generalised grid method

We extend the grid method along three lines.

(i) We construct tilings of any dimension D. For this purpose, we define a D-dimensional grid which is the union of $(D-1)$-dimensional hyperplanes in E^D.

(ii) The grid vectors g_i can be any N vectors in E^D which together span E^D.

(iii) The vectors which span the tiles (the *tiling vectors*) need not to be identical to the grid vectors. With every grid vector g_i we can associate an essentially freely chosen tiling vector t_i. The only restriction is that, for every D-tuple $(i_1, \ldots, i_D)$ of indices, the corresponding D-tuples of grid and tiling vectors, $(g_{i_1}, \ldots, g_{i_D})$ and $(t_{i_1}, \ldots, t_{i_D})$, span a volume of the same orientation. From geometrical considerations in the next section it will become clear that this condition is necessary and sufficient for the tiling not to overlap. This is true at least for dimensions $D \leq 3$, and we conjecture that it holds for any D.

Following these three points, we define an N grid $G_N \subset E^D$ by

$$G_N = \{x \in E^D | x \cdot \hat{g}_l - \gamma_l = k_l |g_l|; l = 1, \ldots, N, k_l \in \mathbb{Z}\} \tag{7}$$

where $\hat{g}_l = g_l / |g_l|$. The orthogonal distance between two neighbouring hyperplanes of

† Note that our terminology differs from that of Kramer and Neri (1984). They use the term 'grid' for $\mathscr{L}$.

the *l*th array is thus $|g_l|$. Again we assume the grid to be regular: there are no points where more than D hyperplanes intersect. As before, we define a vector function $K: E^D \to \mathbb{Z}^N$ by its components:

$$K_i(y) = \lceil |g_i|^{-1} \cdot (y \cdot \hat{g}_i - \gamma_i) \rceil \qquad i = 1, \ldots, N. \tag{8}$$

On a small neighbourhood of an intersection point x_0 of D grid hyperplanes, $K(y)$ takes 2^D different values $K(x_0, j)$, $j = 1, \ldots, 2^D$. The vectors

$$P(x_0, j) = \sum_{i=1}^{N} K_i(x_0, j) \cdot t_i \qquad j = 1, \ldots, 2^D \tag{9}$$

point to the vertices of a D-dimensional parallelepiped. We claim that the parallelepipeds corresponding to all intersection points of the grid form a perfect tiling of E^D, provided the 'non-overlapping condition' is satisfied. This will follow from the correspondence of the above grid method with the generalised projection method presented in the next section.

5. The generalised projection method

The 'classical' projection method is extended with respect to the following three points.

(i) The formalism easily applies to general, not necessarily cubic, lattices. If $\{h_l\}_{l=1,\ldots,N}$ is a basis in E^N, the definition corresponding to (4) is

$$\mathcal{L} = \{x \in E^N \,|\, x \cdot \hat{h}_l - \gamma_l = k_l \cdot |h_l|; \ l = 1, \ldots, N, \ k_l \in \mathbb{Z}\} \tag{10}$$

where $\hat{h}_l = h_l/|h_l|$. The points where N hyperplanes of $\mathcal{L}$ intersect form a lattice L which is generated by that basis $\{b_i\}$ in E^N which spans an elementary cell of $\mathcal{L}$. The strip S around a D-dimensional subspace E^D is still given by (6), but with $\{h_i\}$ replaced by $\{b_i\}$. For orthogonal lattices, $\{b_i\}$ is identical to $\{h_i\}$. Since $\mathcal{L} \cap E^D$ has to be identified with the grid G_N, we call E^D the *grid space* and denote it by E_G^D.

(ii) The unique D-dimensional lattice hypersurface Σ contained in S is projected onto the *tiling space* E_T^D, a D-dimensional subspace of E^N which is not necessarily parallel to E_G^D. The grid space and the tiling space thus have to be carefully distinguished.

(iii) We apply a general regular linear transformation $T: E_T^D \to E_T^D$ to the projected pattern. The tiling thus is obtained by applying a linear transformation A to the hypersurface Σ, where A is the product of the orthogonal projection P onto E_T^D and a subsequent linear transformation T within E_T^D.

The first two points have already been mentioned as possible extensions by Duneau and Katz (1985).

Remark. If we want to avoid overlapping tilings, we are not completely free any more in embedding the tiling space in E^N, once the lattice and the grid space are fixed. We have to embed E_T^D such that the orthogonal projection P of the lattice hypersurface Σ onto E_T^D is an injective mapping. This is the case if it is possible to choose orientations on the hypersurfaces Σ and E_T^D such that P is (everywhere) orientation preserving. The generalised projection method and the non-overlapping condition are visualised in figure 1 for the simple case $N = 2$, $D = 1$. In figure 1(*b*) one sees that for either orientation on Σ there are segments on Σ where P changes the orientation. Bearing this geometrical situation in mind, the reader will easily verify that the non-overlapping condition stated for the generalised grid method is in fact the correct one.
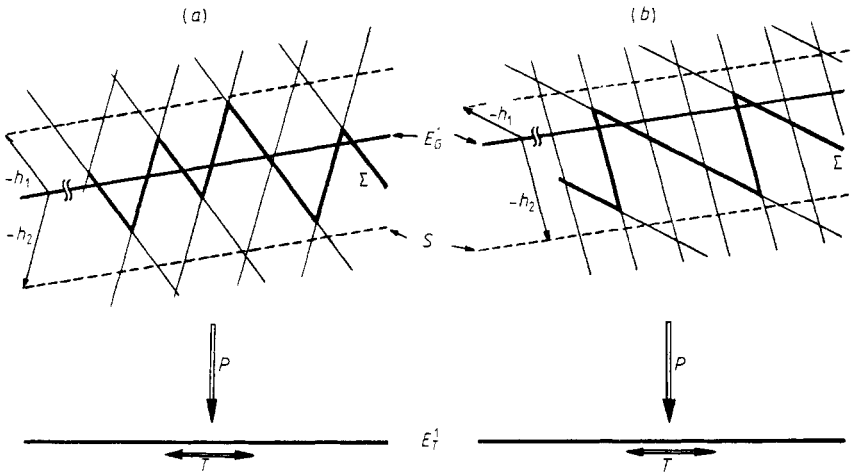
Figure 1. Visualisation of the projection procedure for the simple case $N = 2$, $D = 1$. The non-overlapping condition is satisfied in (a) and violated in (b).

6. The equivalence of the two generalised methods

From the previous arguments it has become clear that every tiling constructed by the generalised projection technique can also be obtained by the generalised grid method. Recall that we have assumed a generic position of E_G^D, i.e. we assume that $\mathscr{L} \cap E_G^D$ is a regular N grid. The extension to singular N grids and grid space positions is straightforward, see e.g. de Bruijn (1981). It remains to show that the converse is true as well.

Let us consider a fixed tiling, defined by a regular N grid G_N and a set of tiling vectors $\{t_i\}_{i=1,\ldots,N}$. We can always find a lattice $\mathscr{L}$ of hyperplanes[†]

$$\mathscr{L} = \{x \in E^N | x \cdot \hat{h}_l - \gamma_l = k_l \cdot |h_l|; \ l = 1, \ldots, N, k_l \in \mathbb{Z}\} \tag{11}$$

such that the grid G_N^D is just the intersection of $\mathscr{L}$ with some D-dimensional subspace of E^N. This subspace thus has to be identified with the grid space E_G^D. In (11) we assume, of course, that the vectors $\{h_i\}_{i=1,\ldots,N}$ generating $\mathscr{L}$ are linearly independent.

Note that the vectors h_i and the grid vectors g_i have to be enumerated consistently. The vector h_l points from one hyperplane of $\mathscr{L}$ to the next in the same array, and so it should be g_l which connects the intersections of these two hyperplanes with the grid space E_G^D.

The strip S around E_G^D, containing a unique D-dimensional lattice hypersurface Σ, is again given by (6), but with h_i replaced by b_i. The main point is now to find the correct embedding of the tiling space, containing the tiling vectors, in E^N. (The tiling vectors t_i are supposed to be fixed within E_T^D.) In general, we can neither hope that it is possible to choose the grid and the tiling spaces to be the same nor that the tiling vectors are the projections of the lattice vectors b_i onto the tiling space. This can easily be derived from the results of Hadwiger (1940) (the same arguments can be found in Coxeter (1973)) or from a simple counterexample for $N = 2$, $D = 1$. Therefore, let us consider a 'preliminary' (arbitrary) embedding $\tilde{E}_T^D \subset E^N$ of the tiling space, and denote

[†] Note that the choice of this lattice is not unique. In fact, there is a huge variety of possibilities to choose $\mathscr{L}$.

by $\tilde{t}_i$ the corresponding tiling vectors. We define a linear transformation $\tilde{A}: E^N \to \tilde{E}_T^D \subset E^N$ by its action on the basis $\{b_i\}_{i=1,\dots,N}$:

$$\tilde{A}b_i = \tilde{t}_i \qquad i = 1, \dots, N. \tag{12}$$

Since the orthogonal complement of the kernel of $\tilde{A}$, $(\ker \tilde{A})^\perp$, and the preliminary tiling space $\tilde{E}_T^D$ have the same dimension, there exists an orthogonal transformation $R: E^N \to E^N$, which transforms $\tilde{E}_T^D$ onto $(\ker \tilde{A})^\perp$. The action of R on the orthogonal complement of $\tilde{E}_T^D$ is irrelevant for our purposes. The 'final' tiling space is now identified with $(\ker \tilde{A})^\perp$, $E_T^D \equiv (\ker \tilde{A})^\perp$. The linear transformation $A \equiv R \circ \tilde{A}$ maps the lattice hypersurface Σ onto E_T^D. In particular, the tiling vectors are given by

$$t_i = Ab_i \equiv R \circ \tilde{A}b_i \qquad i = 1, \dots, N.$$

Since the kernels of A and $\tilde{A}$ are the same, E_T^D and $\ker A$ are orthogonal. Hence, A has the property

$$A = A \circ P = (A \upharpoonright E_T^D) \circ P \tag{13}$$

where P is the orthogonal projection onto E_T^D and $A \upharpoonright E_T^D$ denotes the restriction of A to E_T^D. Therefore, we can identify

$$T \equiv A \upharpoonright E_T^D \tag{14}$$

which completes the proof that the two generalised methods are equivalent.

We close this section with some remarks.

(i) The independence of the grid vectors and the tiling vectors allows us to fix the topology of the tiling and the shapes of the tiles independently from each other. In the projection formalism, this corresponds to the freedom of choosing the grid space and the tiling space independently.

(ii) The lattice $\mathscr{L}$ in E^N is not uniquely specified by the grid. One may ask therefore whether, given a grid and a set of tiling vectors t_i, there exists a special choice for the lattice, for which E_T^D can be embedded in such a way that the t_i are just orthogonal projections of the lattice vectors b_i, i.e. $T = \mathbb{1}$. It is easy to find examples in the simple case $D = 1$, $N = 2$, where this is impossible. However, we do not know the general conditions one has to impose on the grid and the tiling vectors in order to reduce A to a bare projection ($T = \mathbb{1}$).

(iii) The question whether or not the tiling has any translational symmetry can easily be answered within the projection formalism. Since the tiling is a linear transformation of some subset of the strip S, it has a translational symmetry if and only if this is the case for S. This in turn is true if and only if the grid space E_G^D contains a lattice vector u. In formal terms, every lattice vector $u \in L$ contained in E_G^D leads to a translational symmetry Au of the tiling (for examples see § 8).

An interesting insight can be gained by fixing the lattice $\mathscr{L}$, the tiling space E_T^D and the transformation T, but allowing the grid space E_G^D to vary. By varying E_G^D we change the topology of the tiling, while the shapes of the tiles are kept fixed. If we keep the change in the orientation of E_G^D small enough, an arbitrary large portion of the tiling containing the origin is left unchanged. In particular, since there are always large lattice vectors almost parallel to E_G^D, by a slight change of E_G^D we can turn a non-periodic tiling into a periodic one with a large unit cell, keeping fixed a considerable portion around the origin. Hence, by looking at a finite piece of a tiling, we can never decide whether it is part of a truly non-periodic tiling or of a periodic one with a large unit cell.

(iv) Further generalisations of the grid method exist. The hyperplanes of the grid G_N^D need not be arranged periodically; the grid method, with a suitably defined vector function $K(y)$, still yields perfect tilings. The correspondence to the projection framework is somewhat more complicated, however, and we do not discuss it here. We just stress that the tiling cannot be obtained by simply arranging the hyperplanes of the lattice $\mathscr{L}$ aperiodically. The aperiodic lattice, when transformed by A, would lead to a pattern consisting of tiles whose shapes depend on their location. This is obviously not the case for the tiles generated by the grid method, whose shapes are defined by fixed tiling vectors.

7. Fourier transformation

The particular properties (13) and (14) of the linear transformation A make it easy to apply the recently developed methods for the evaluation of Fourier spectra. A is the product of an orthogonal projection $P: (S \cap L) \to E_T^D$ and a linear transformation $T: E_T^D \to E_T^D$. We therefore subdivide the calculation into two steps. We first determine the Fourier transform of the *projected* pattern and then include the effect of the transformation T.

For the following, it is convenient to use the orthogonal decomposition $E^N = E_T^D \oplus E_{T,\perp}^D$. For each $x \in E^N$, we write $x = x_{\parallel} + x_{\perp}$, where $x_{\parallel} \in E_T^D$ and $x_{\perp} \in E_{T,\perp}^D$. In order to make the notation simple, we use the density of lattice points,

$$\rho_L(x) = \sum_{j \in \mathbb{Z}^N} \delta\left(x - \sum_{l=1}^N j_l b_l\right)$$

instead of the lattice L itself (see definition (5)). Following Zia and Dallas (1985), we write the projected density as

$$Q(x_{\parallel}) = \int d^{N-D}x_{\perp} C_S(x_{\parallel}, x_{\perp}) \cdot \rho_L(x_{\parallel}, x_{\perp})$$

$$= \int d^{N-D}k'_{\perp} \tilde{C}_S(x_{\parallel}, -k'_{\perp}) \cdot \rho_L(x_{\parallel}, k'_{\perp})$$

where $C_S(x_{\parallel}, x_{\perp})$ is the characteristic function of the strip S, and the tilde denotes the Fourier transform with respect to $x_{\perp}$. In contrast to the case of Zia and Dallas (1985), C_S is $x_{\parallel}$ dependent because E_T^D and S are not parallel here. The Fourier transform $\hat{Q}(k_{\parallel})$ of $Q(x_{\parallel})$ is just a convolution:

$$\hat{Q}(k_{\parallel}) = \int d^D k'_{\parallel} \int d^{(N-D)}k'_{\perp} V_S(k_{\parallel} - k'_{\parallel}, -k'_{\perp}) \cdot \tilde{\rho}_L(k'_{\parallel}, k'_{\perp})$$

$$= \int d^N k' V_S(k_{\parallel} - k'_{\parallel}, -k'_{\perp}) \cdot \rho_{\bar{L}}(k') \tag{15}$$

where $k' = k'_{\parallel} + k'_{\perp}$. V_S is the Fourier transform of C_S and $\rho_{\bar{L}}(k)$ the Fourier transform of $\rho_L(x)$, i.e. the density of points in the reciprocal lattice $\bar{L}$. Therefore $\hat{Q}(k_{\parallel})$ is a weighted sum of δ functions.

Now the transformation T can easily be incorporated. We denote the transformed density by $Q_T(x_\parallel)$; after some algebra, we find

$$Q_T(x_\parallel) = \frac{1}{|\det T|} Q(T^{-1}x_\parallel) \tag{16}$$

and

$$\hat{Q}_T(k_\parallel) = \hat{Q}(T^+k_\parallel) \tag{17}$$

where T^+ is the adjoint of T. Inserting this result into (15), we get the Fourier transform $\mathcal{F}(k_\parallel)$ of the final pattern:

$$\mathcal{F}(k_\parallel) \equiv \hat{Q}(T^+k_\parallel) = \int d^N k' V_S(T^+k_\parallel - k'_\parallel, -k'_\perp) \cdot \rho_{\tilde{L}}(k'). \tag{18}$$

8. Symmetric two-dimensional grids

In this section, we discuss the special class of 2D tilings generated by grids whose grid vectors $\{g_i\}_{i=1,\ldots,N}$ form an N-fold symmetric star, i.e. point to the vertices of a regular N-gon. The tiling vectors t_i are assumed to have the same directions as the grid vectors. For later convenience, we choose the normalisations

$$g_i = (N/2)^{1/2} e_i \qquad t_i = (2/N)^{1/2} e_i \tag{19}$$

(with the unit vectors e_i forming an N-fold symmetric star).

We now describe how to obtain these tilings by the projection method. As it turns out, for our choice of grid and tiling vectors it is possible to use a cubic lattice

$$\mathcal{L} = \{x \in E^N | (x - \gamma) \cdot h_l = k_l; \ k_l = 1, \ldots, N, \ l \in \mathbb{Z}\} \tag{20}$$

where the components of γ are the grid parameters γ_i and $\{h_l\}_{l=1,\ldots,N}$ is an orthonormal basis of E^N. The high symmetry of the problem allows the application of powerful group theoretic methods. The point group of $\mathcal{L}$ is the hyperoctahedral group $\Omega(N)$. Its natural representation D^N is reducible under subduction to the subgroup $C_N \subset \Omega(N)$ of all cyclic permutations of the basis vectors of the lattice. Its decomposition depends on whether N is even or odd:

$$D^N \xrightarrow[\Omega(N) \to C_N]{} \begin{cases} D_1^1 \oplus \sum\limits_{i=1}^{[N/2]} D_i^2 & N \text{ odd} \\[4mm] D_1^1 \oplus D_{n/2}^1 \oplus \sum\limits_{i=1}^{N/2-1} D_i^2 & N \text{ even.} \end{cases} \tag{21}$$

(The upper index denotes the dimension of the representations; $[n/2]$ is the largest integer smaller than or equal to $n/2$.) The invariant subspace U^2 of one of the 2D representations is spanned by the basis

$$v_1 = \left(\frac{2}{N}\right)^{1/2} \sum_{l=1}^N h_l \cdot \cos\left(\frac{2\pi}{N}(l-1)\right) \qquad v_2 = \left(\frac{2}{N}\right)^{1/2} \sum_{l=1}^N h_l \cdot \sin\left(\frac{2\pi}{N}(l-1)\right). \tag{22}$$

Starting from de Bruijn's arguments for the special case $N = 5$ it is straightforward to show that the grid construction corresponds to the orthogonal projection P of $(L \cap S)$ onto U^2 (here S is the strip around U^2 defined by (6) and (22)). It is easy to show that the grid and tiling vectors that correspond to our projection construction are

indeed given by (19). In the language of the generalised grid method we have therefore

$$A = P \qquad \text{i.e. } T = 1$$
$$E_T^2 = E_G^2 = U^2. \tag{23}$$

From remark (iii) in § 6, it follows that the tilings have a translational symmetry Pu if there exists a lattice vector $u \in L$ and real numbers r and s such that

$$u = \left(\frac{N}{2}\right)^{1/2} (r v_1 + s v_2) = \sum_{l=1}^{N} h_l \cdot \left[r \cos\left(\frac{2\pi}{N}(l-1)\right) + s \sin\left(\frac{2\pi}{N}(l-1)\right) \right]. \tag{24}$$

The condition that $u \in U^2$ is a lattice vector means that

$$r \cos\left(\frac{2\pi}{N}(l-1)\right) + s \sin\left(\frac{2\pi}{N}(l-1)\right) \in \mathbb{Z} \; \forall l. \tag{25}$$

It is easy to see that this can only be satisfied for $N = 2, 3, 4, 6$, which are just the allowed crystal symmetries.

Obviously, the generalised grid method allows the construction of 2D quasiperiodic patterns with *any point symmetry*. This is in contrast to the results of Levine and Steinhardt (1984). Using a different construction, they get, besides the allowed crystal symmetries, a rather restricted set of possible point symmetries ($N = 8$, P or $2P$, where P is a prime number). The difference may be due to additional conditions imposed by Levine and Steinhardt, e.g. self-similarity.

The number M_N of different tiles (where we do not distinguish between different orientations) of a tiling generated by a symmetric N grid and a symmetric star of tiling vectors is easily found to be

$$M_N = \begin{cases} [N/2] & N \text{ odd} \\ [N/4] & N \text{ even} \end{cases} \tag{26}$$

where again $[x]$ is the largest integer smaller or equal to x. Interestingly enough, M_N coincides with the number of incommensurate intervals Levine and Steinhardt (1984) need in their alternative construction of quasiperiodic lattices.

In figure 2 we present a tiling exhibiting 12-fold symmetry. Its tiles are a square and two rhombuses, one with a 60° and one with a 30° angle. This tiling looks strikingly similar to electron microscope images of a new phase of Ni–Cr alloy recently discovered (Ishimasa *et al* 1985).

9. Quasiperiodic tilings with periodic inclusions

As a final application we discuss a new principle for the generation of quasiperiodic patterns containing arbitrarily large periodic regions. We illustrate this principle with a simple example. Take, e.g. a 5-grid, in which two arrays of grid lines are narrowed with respect to the remaining ones, as shown in figure 3(*a*). We observe that

(i) to all intersection points of two given grid arrays, there corresponds a unique rhombus having a unique orientation;

(ii) the rhombuses corresponding to two adjacent intersection points have one common edge (by 'adjacent', we mean that the two intersection points are not separated by any grid line).

Therefore the intersection points lying in a region bounded by thin grid lines (see figure 3(a)) give rise to rhombuses that are arranged periodically. Note, however, that 5-fold symmetry is destroyed by this procedure. An example is shown in figure 3(b).
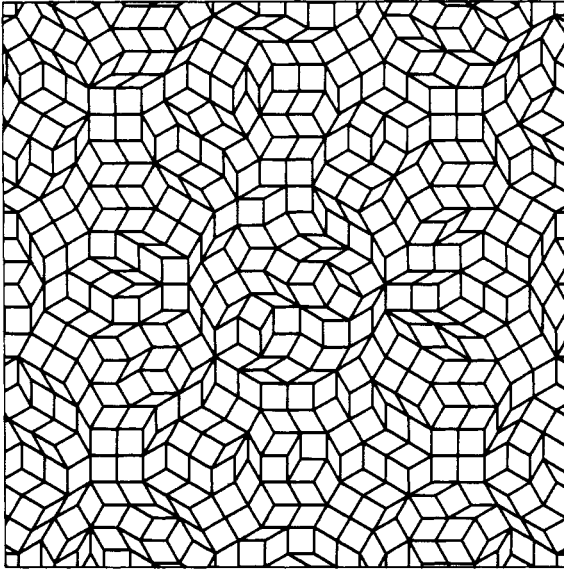
Figure 2. Pattern with 12-fold bond orientational symmetry. Its Fourier spectrum consists of a 12-fold symmetric arrangement of δ peaks filling reciprocal space densely.

(a)   (b)
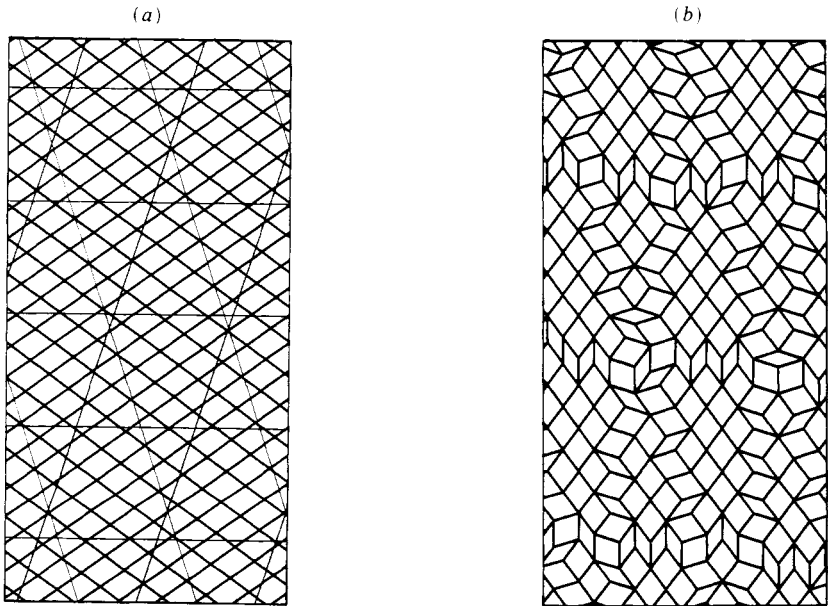
Figure 3. (a) Example of a grid yielding periodic inclusions. The areas bounded by the thin grid lines generally contain a lot of intersection points of the same type, leading to adjacent tiles of the same type. (b) Tiling with periodic inclusions. The tiles are the Penrose rhombuses.

Acknowledgments

Note added. Towards the end of the preparation of this paper we received a very interesting preprint from Socolar *et al* (1985). They also consider generalisations of de Bruijn's method. The connection to the projection formalism is not discussed however.

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